Borel classes dimensions(General Topology, Geometric Topology and Their Applications)

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1 Introduction and results.

The classes of topological spaces are assumed to be

1. non-empty (we suppose that at least the empty space $\emptyset$ is a member), and

2. monotone with respect to closed subsets.

The letter $\mathcal{P}$ is used to denote a such class and the following classes of spaces satisfiy the conditions 1 and 2 above.

- The class of compact metrizable spaces $\mathcal{K}$.

- The class of σ-compact metrizable spaces $\mathcal{S}$.

- The class of completely metrizable spaces $\mathcal{C}$.

- The class of separable completely metrizable spaces $\mathcal{C}_0$.

Let X be a space and A, B disjoint subsets of X. We recall that a closed set $C \subset X$ is said to be a *partition* between A and B in X if there are disjoint open subsets U and V of X such that $A \subset U$, $B \subset V$ and $C = X \setminus (U \cup V)$.

In [4] Lelek introduced the small inductive dimension modulo a class $\mathcal{P}$, $\mathcal{P}$-ind, which is a natural generalization of well known dimension functions such as the small inductive dimension ind and the small inductive compactness degree cmp.

Definition 1.1 Let X be a regular T_1-space and $\mathcal{P}$ a class of spaces. Then we define the *small inductive dimension modulo a class* $\mathcal{P}$, $\mathcal{P}\text{-ind}\, X$, of X as follows.

(i) $\mathcal{P}\text{-ind}\, X = -1$ iff $X \in \mathcal{P}$.

(ii) For a natural number n, $\mathcal{P}\text{-ind}\, X \leq n$ if for any point $x \in X$ and any closed subset A of X with $x \notin A$ there exists a partition C between x and A in X such that $\mathcal{P}\text{-ind}\, C < n$.

The small inductive dimension modulo a class $\mathcal{P}$ has a natural transfinite extension.

Definition 1.2 Let X be a regular T_1-space and α either an ordinal number or the integer -1. Then the *small transfinite inductive dimension modulo* $\mathcal{P}$, $\mathcal{P}\text{-trind}\, X$, of X is defined as follows.

(i) $\mathcal{P}\text{-trind}\, X = -1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}\text{-trind}\, X \leq \alpha$ if for any point $x \in X$ and any closed subset A of X with $x \notin A$ there exists a partition C between x and A in X such that $\mathcal{P}\text{-trind}\, C < \alpha$.

(iii) $\mathcal{P}\text{-trind}\, X = \alpha$ if $\mathcal{P}\text{-trind}\, X \leq \alpha$ and $\mathcal{P}\text{-trind}\, X > \beta$ for any ordinal $\beta < \alpha$;

(iv) $\mathcal{P}\text{-trind}\, X = \infty$ if $\mathcal{P}\text{-trind}\, X > \alpha$ for any ordinal α.

We notice the following.

- $\{\emptyset\}\text{-trind}\, X = \text{trind}\, X$, i.e., the small transfinite dimension.

- $\mathcal{K}$-ind $X = \operatorname{cmp} X$ (and $\mathcal{K}$-trind $X = \operatorname{trcmp} X$), i.e., the small (transfinite) compactness degree.

- $\mathcal{C}$-ind $X = \operatorname{icd} X$ (and $\mathcal{C}$-trind $X = \operatorname{tricd} X$), i.e., the small (transfinite) completeness degree.

- If $\mathcal{P}_2 \subset \mathcal{P}_1$, then $\mathcal{P}_1$-trind $X \leq \mathcal{P}_2$-trind X; in particular, $\operatorname{tricd} X \leq \operatorname{trcmp} X \leq \operatorname{trind} X$ holds.

Here, we shall consider on the absolute Borel classes. For each ordinal number α, let $\mathcal{A}(\alpha)$ and $\mathcal{M}(\alpha)$ be the *absolute additive class* α and the *absolute multiplicative classe* α, respectively. Further, $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$ is said to be the *absolute ambiguous class* α and we write $\mathcal{AB} = \bigcup\{\mathcal{A}_\alpha : \alpha < \omega_1\}$. We notice that the absolute Borel classes in the universe of metrizable spaces satisfy the conditions 1 and 2.

Recall that in the universe of separable metrizable spaces, we have the following.

- $\mathcal{A}(0) = \{\emptyset\}$.
- $\mathcal{M}(0) = \mathcal{K}$.
- $\mathcal{A}(1) = \mathcal{S}$.
- $\mathcal{M}(1) = \mathcal{C}_0$.
- A diagram of the hierarchy of absolute Borel classes:

$$\begin{array}{ccccccc}
 & & \mathcal{A}(1)=\mathcal{S} & & & \mathcal{A}(2) & \cdots \\
 & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\
\{\emptyset\} \subseteq \mathcal{K} \subseteq \mathcal{A}(1)\cap\mathcal{M}(1) & & & & \mathcal{A}(2)\cap\mathcal{M}(2) & & \cdots \\
 & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\subsetneq$} & \\
 & & \mathcal{M}(1)=\mathcal{C}_0 & & & \mathcal{M}(2) & \cdots
\end{array}$$

We have a trivial example which shows the difference between trind and trcmp: The Hilbert cube $\mathbb{I}^\infty$ has $\operatorname{trind} \mathbb{I}^\infty = \infty$ and $\operatorname{cmp} \mathbb{I}^\infty (= \operatorname{icd} \mathbb{I}^\infty = \mathcal{S}\text{-ind}\, \mathbb{I}^\infty) = -1$. Furthermore, E. Pol constructed the following example.

Example 1.1 (E. Pol, [5]) There exists a σ-compact, completely metrizable space P such that $\operatorname{trcmp} P = \infty$ (i.e., $\operatorname{trind} P = \operatorname{trcmp} P = \infty$ and $\operatorname{tricd} P = \mathcal{S}\text{-}\operatorname{trind} P = \mathcal{A}(1) \cap \mathcal{M}(1)\text{-}\operatorname{trind} P = -1$).

Thus, we may ask whether we can generalize Pol's example to every ordinal number $\alpha < \omega_1$.

It is well know that the small compactness degree cmp is related to an extension property, i.e., de Groot proved that a separable metrizable space X is rim-compact (i.e., $\operatorname{cmp} X \leq 0$) iff X has a metric compactification Y such that $\dim(Y - X) \leq 0$. Connect with this theorem, we introduce other two dimension-like functions.

Definition 1.3 Let $\mathcal{P}$ be a class of spaces. We recall that a separable metrizable space Y is a *$\mathcal{P}$-hull* (resp. *$\mathcal{P}$-kernel*) of a separable metrizable space X if $Y \in \mathcal{P}$ and $X \subset Y$ (resp. $Y \subset X$). Then the *small transfinite $\mathcal{P}$-deficiency*, $\mathcal{P}\text{-}\operatorname{trdef} X$, and the *small transfinite $\mathcal{P}$-surplus*, $\mathcal{P}\text{-}\operatorname{trsur} X$, of a separable metrizable space X are defined by

$$\mathcal{P}\text{-}\operatorname{trdef} X = \min\{\operatorname{trind}(Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\},$$

$$(\mathcal{P}\text{-}\operatorname{def} X = \min\{\operatorname{ind}(Y \setminus X) : Y \text{ is an } \mathcal{P}\text{-hull of } X\}),$$

$$\mathcal{P}\text{-}\operatorname{trsur} X = \min\{\operatorname{trind}(X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\},$$

$$(\mathcal{P}\text{-}\operatorname{sur} X = \min\{\operatorname{ind}(X \setminus Y) : Y \text{ is an } \mathcal{P}\text{-kernel of } X\}).$$

It is clear that the functions $\mathcal{P}$-trdef and $\mathcal{P}$-trsur are transfinite extensions of the functions $\mathcal{P}$-def and $\mathcal{P}$-sur, respectively, which are discussed in [1]. It is also clear that if $\mathcal{P}_2 \subset \mathcal{P}_1$, then $\mathcal{P}_1\text{-}\operatorname{trdef} X \leq \mathcal{P}_2\text{-}\operatorname{trdef} X$ and $\mathcal{P}_1\text{-}\operatorname{trsur} X \leq \mathcal{P}_2\text{-}\operatorname{trsur} X$.

Recall also that for the function $\mathcal{K}$-def is the well known compact deficiency def. We will denote the transfinite extension $\mathcal{K}$-trdef of the compact deficiency def by trdef.

Facts (cf. [1]). Let X be a separable metrizable space and α an ordinal number. Then we have the following.

1. If $\alpha = 0$, then $\mathcal{M}(0)$-ind $X \le \mathcal{M}(0)$-def $X \le \mathcal{M}(0)$-sur X holds and the converse of the inequalities do not hold. (We notice that $\mathcal{M}(0) = \mathcal{K}$ and so $\mathcal{M}(0)$-ind X = cmp X and $\mathcal{M}(0)$-def X = def X.) We also notice that $\mathcal{A}(0) = \{\emptyset\}$ and hence $\mathcal{A}(0)$-ind $X = \mathcal{A}(0)$-sur X trivially holds and $\mathcal{A}(0)$-def X can not be defined if $X \neq \emptyset$.

2. If $\alpha = 1$, then $\mathcal{A}(1)$-ind $X \le \mathcal{A}(1)$-def $X = \mathcal{A}(1)$-sur X and $\mathcal{M}(1)$-ind $X = \mathcal{M}(1)$-def $X \le \mathcal{M}(1)$-sur X hold. The converses of the inequalities above do not hold. (We notice that $\mathcal{A}(1) = \mathcal{S}$ and $\mathcal{M}(1) = \mathcal{C}_0$ and so $\mathcal{M}(1)$-ind X = icd X.)

3. If $\alpha \ge 2$, then $\mathcal{A}(\alpha)$-ind $X = \mathcal{A}(\alpha)$-def $X = \mathcal{A}(\alpha)$-sur X and $\mathcal{M}(\alpha)$-ind $X = \mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-sur X hold.

M. Charalambous [2] showed that the equality $\mathcal{M}(\alpha)$-def $X = \mathcal{M}(\alpha)$-ind X can not be extended to the transfinite dimension for the case of $\alpha = 1$.

Example 1.2 (M. Charalambous, [2]) There exists a separable metrizable space C such that $\mathcal{C}$-trdef C (= $\mathcal{M}(1)$-trdef C) $= \omega_0$ and tricd C (= $\mathcal{M}(1)$-trind C) $= \infty$. (We notice that $\mathcal{C}_0$-trdef $\le$ tricd X holds for every separable metrizable space.)

Thus, it seems to be natural that we ask whether for each ordinal number $\alpha < \omega_1$ there exits a separable metrizable space X such that $\mathcal{M}(\alpha)$-trdef $X = \omega_0$ and $\mathcal{M}(\alpha)$-trind $X = \infty$ or $\mathcal{A}(\alpha)$-trdef $X = \omega_0$ and $\mathcal{A}(\alpha)$-trind $X = \infty$.

Connect with the questions above, we have the following.

Theorem 1.1 *Let α be any ordinal with $1 \le \alpha < \omega_1$.*
(1) There exist separable metrizable spaces X_α, Y_α and Z_α such that

(a) $f X_\alpha$, $f Y_\alpha$, $f Z_\alpha \le \omega_0$, *where f is either* trdef *or* $\mathcal{K}$-trsur;

(b) $\mathcal{M}(\alpha)$-trind $X_\alpha = -1$ *and* $\mathcal{A}(\alpha)$-trind $X_\alpha = \infty$ *(and hence* $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha = \infty$*);*

(c) $\mathcal{A}(\alpha)$-trind $Y_\alpha = -1$ *and* $\mathcal{M}(\alpha)$-trind $Y_\alpha = \infty$ *(and hence* $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$-trind $X_\alpha = \infty$*);*

(d) $\mathcal{M}(\alpha)$-trind $Z_\alpha = \mathcal{A}(\alpha)$-trind $Z_\alpha = \infty$ *and* $\mathcal{A}(\alpha + 1) \cap \mathcal{M}(\alpha + 1)$-trind $Z_\alpha = -1$.

(2) There does not exist a separable metrizable space W_α such that $\mathcal{A}(\alpha)$-$\operatorname{trind} W_\alpha \neq \infty$*, $\mathcal{M}(\alpha)$-*$\operatorname{trind} W_\alpha \neq \infty$ *and $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$-*$\operatorname{trind} W_\alpha = \infty$.

Theorem 1.2 *There exists a separable metrizable space X with* $\operatorname{trdef} X = \mathcal{K}$-$\operatorname{trsur} X = \omega_0$ *such that for each $1 \leq \alpha < \omega_1$ we have $\mathcal{B}$-*$\operatorname{trind} X = \infty$ *and $\mathcal{B}$-*$\operatorname{trdef} X = \mathcal{B}$-$\operatorname{trsur} X = \omega_0$*, where $\mathcal{B} = \mathcal{A}(\alpha), \mathcal{M}(\alpha)$ or $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)$.*

Remark 1.1 By Thereoms 1.1 and 1.2, it follows that the equalities $\mathcal{M}(\alpha)$-$\operatorname{def} X = \mathcal{M}(\alpha)$-$\operatorname{ind} X$ and $\mathcal{A}(\alpha)$-$\operatorname{sur} X = \mathcal{A}(\alpha)$-$\operatorname{ind} X$ can not be extended to transfinite-dimensional cases. For the spaces X_α, $Y_{|alpha}$ and Z_α in Theorem 1.1, we additionally have that

- $\mathcal{M}(\alpha)$-$\operatorname{trdef} X_\alpha = \mathcal{A}(\alpha)$-$\operatorname{trsur} Y_\alpha = -1$;
- $\mathcal{M}(\alpha)$-$\operatorname{trdef} Y_\alpha = \mathcal{M}(\alpha)$-$\operatorname{trdef} Z_\alpha = \mathcal{A}(\alpha)$-$\operatorname{trsur} X_\alpha = \mathcal{A}(\alpha)$-$\operatorname{trsur} Z_\alpha = \omega_0$.

We refer the readers to the books [1], [3] and [7] for the dimensions modulo classes, dimension theory and the theory of Borel sets, respectively.

2 Outline of proofs.

All classes of topological spaces considered here are additionally assumed to be finitely additive. We will follow some idea of E. Pol [5]. Let $\mathcal{P}$ be a class of topological spaces. A space X is said to have the *property* $(*)_\mathcal{P}$ if for every sequence $\{(A_i, B_i)\}_{i=1}^\infty$ of pairs of disjoint compact subsets of X there exist partitions L_i between A_i and B_i in X and an integer N such that $\cap_{i=1}^N L_i \in \mathcal{P}$.

It is evident that the property $(*)_\mathcal{P}$ is closed hereditary.

We have two propositions on the property $(*)_\mathcal{P}$.

Proposition 2.1 *If a space X is covered by a finite family of closed sets such that each element of this cover possesses property $(*)_\mathcal{P}$ then X also possesses this property.*

Proposition 2.2 *Let X be a space. If $\mathcal{P}$-*$\operatorname{trind} X \neq \infty$ *then X possesses property $(*)_\mathcal{P}$.*

Let $\mathbb{I}^\infty = \{(x_i) : 0 \le x_i \le 1, i = 1,2,...\}$ be the Hilbert cube and $Z = \{0, \frac{1}{2}, \frac{1}{3}, ...\}$ a subspace of the unit interval $\mathbb{I}$. For each $n \ge 1$ we denote the subset $\{(x_i) \in \mathbb{I}^\infty : x_j = 0 \text{ for } j \ge n+1\}$ of $\mathbb{I}^\infty$ by $\mathbb{I}^n$. For each $n \ge 1$ and each $i = 1, ..., n$, we put

$$A_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 0\},\ B_i^n = \{(x_i) \in \mathbb{I}^n \subset \mathbb{I}^\infty : x_i = 1\}.$$

Choose for each $n \ge 1$ a subset E_n of $\mathbb{I}^n$ and put

$$X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times E_n). \tag{1}$$

Furthermore, we put $Y = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times \mathbb{I}^n)$. It is clear that $X \subset Y \subset Z \times \mathbb{I}^\infty$, Y is compact, and $Y \setminus X$ is a subspace of the topological sum $\oplus_{n=1}^{\infty} \mathbb{I}^n$. Thus, $\operatorname{trind}(Y \setminus X) \le \omega_0$. Observe also that $\operatorname{trind}(X \setminus (\{0\} \times \mathbb{I}^\infty)) \le \omega_0$. Hence

$$\operatorname{trdef} X \le \omega_0 \text{ and } \mathcal{K}\text{-trsur} X \le \omega_0. \tag{2}$$

Lemma 2.1 *If for each $m \ge 1$ there exists an integer $k(m) \ge m+1$ such that for any $n \ge k(m)$ and any partition L_i^n between A_i^n and B_i^n in $\mathbb{I}^n, i \le m$, we have $E_n \cap \bigcap_{i=1}^m L_i^n \notin \mathcal{P}$, then $\mathcal{P}$-$\operatorname{trind} X = \infty$.*

Proof. By Proposition 2.2, it suffices to show that X does not have the property $(*)_{\mathcal{P}}$. For each $i = 1, 2, \ldots$ let L_i be a partition between compact sets $A_i = \{(0,(x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 0\}$ and $B_i = \{(0,(x_j)) \in \{0\} \times \mathbb{I}^\infty : x_i = 1\}$ We shall show that $\cap_{i=1}^N L_i \notin \mathcal{P}$ for every natural number N. Let N be a natural number. For each $i \ge 1$ let us consider a partition L_i' between A_i and B_i in Y such that $L_i = L_i' \cap X$. Note that for every i there exists a natural number $n_i \ge 2$ such that for any $n \ge n_i$ $L_i^n = L_i' \cap (\{\frac{1}{n}\} \times \mathbb{I}^n)$ is a partition between $\{\frac{1}{n}\} \times A_i^n$ and $\{\frac{1}{n}\} \times B_i^n$ in $\{\frac{1}{n}\} \times \mathbb{I}^n$. Let n a fixed integer with $n \ge \max\{n_1, ..., n_N, k(N)\}$. Then $C = (\bigcap_{i=1}^N L_i^n) \cap (\{\frac{1}{n}\} \times E_n) = (\bigcap_{i=1}^N L_i) \cap (\{\frac{1}{n}\} \times E_n)$ is a closed subset of $\cap_{i=1}^N L_i$, and $C \notin \mathcal{P}$ by the assumption. So $\cap_{i=1}^N L_i \notin \mathcal{P}$.

We shall also use the following.

Lemma 2.2 *([8, Lemma 5.2]) Let L_{i_j} be partitions between the opposite faces $A_{i_j}^n$ and $B_{i_j}^n$ in $\mathbb{I}^n$, where $1 \le i_1 < i_2 ... < i_p \le n$ and $1 \le p < n$. Then for any $k \ne i_j, j = 1, ..., p$, there is a continuum $C \subset \cap_{j=1}^p L_{i_j}$ meeting the faces A_k^n and B_k^n.*

Lemma 2.3 *Let α be an ordinal number with $1 \leq \alpha < \omega_1$. Then there exist subsets Q_α, P_α and D_α of $\mathbb{I}$ such that*

1. $Q_\alpha \in \mathcal{A}(\alpha) - \mathcal{M}(\alpha)$,
2. $P_\alpha \in \mathcal{M}(\alpha) - \mathcal{A}(\alpha)$,
3. $D_\alpha \in \mathcal{A}(\alpha+1) \cap \mathcal{M}(\alpha+1) - (\mathcal{A}(\alpha) \cup \mathcal{M}(\alpha))$.

Proof of Theorem 1.1. *(1)* We shall prove for Y_α only. We put

$$Y_\alpha = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=2}^{\infty} \{\frac{1}{n}\} \times \pi_n^{-1}(Q_\alpha)),$$

where Q_α is the subspace $\mathbb{I}$ described in Lemma 2.3 and $\pi_n : \mathbb{I}^n \to \mathbb{I}$ be the projection onto the n-th factor. By the construction of Y_α, it is clear that $\mathcal{M}(\alpha)$-trdef $Y_\alpha \leq$ trdef $Y_\alpha \leq \omega_0$, and $\mathcal{M}(\alpha)$-trsur $Y_\alpha \leq \omega_0$. Since the absolute Borel classes are preserved under perfect preimages, it follows that $\pi_n^{-1}(Q_\alpha) \in \mathcal{A}(\alpha)$. Thus, $Y_\alpha \in \mathcal{A}(\alpha)$ and hence $\mathcal{A}(\alpha)$-trind $Y_\alpha = -1$. Now, it suffices to show that $\mathcal{M}(\alpha)$-trind $Y_\alpha = \infty$. To apply Lemma 2.1, for every natural number m let $k(m) = m + 1$. For each $n \geq k(m)$ and each $i \leq n$ let L_i^n be a partition between A_i^n and B_i^n in $\mathbb{I}^n$. By Lemmea 2.2, there exsits a continuum C such that $C \subset \bigcap_{i=1}^n L_i^n$ and $C \cap A_i^n \neq \emptyset \neq C \cap B_i^n$. Let $\pi_n^C = \pi|C : C \to \mathbb{I}$ be the restriction of the projection π_n over C. Then $C \cap \pi_n^{-1}(Q_\alpha) = (\pi_n^C)^{-1}(Q_\alpha) \subset \bigcap_{i=1}^n L_i^n \cap \pi_n^{-1}(Q_\alpha)$. Since $C \cap \pi_n^{-1}(Q_\alpha)$ is closed set of $\bigcap_{i=1}^n L_i^n \cap \pi_n^{-1}(Q_\alpha)$ and $(\pi_n^C)^{-1}(Q_\alpha) \notin \mathcal{M}(\alpha)$, it follows that $\bigcap_{i=1}^n L_i^n \cap \pi_n^{-1}(Q_\alpha) \notin \mathcal{M}(\alpha)$. Thus, it follows from Lemma 2.1 that $\mathcal{M}(\alpha)$-trind $Y_\alpha = \infty$. This completes the proof.

(2) The second part of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 2.3 *Let X be a separable metrizable space with $\mathcal{A}(\alpha)$-trind $X \leq \mu_1$ and $\mathcal{M}(\alpha)$-trind $X \leq \mu_2$. Then*

$$\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)\text{-trind}\, X = \begin{cases} \mu_1 + n(\mu_2) + 1, & \text{if } \lambda(\mu_1) = \lambda(\mu_2), \\ \mu_1, & \text{if } \lambda(\mu_1) > \lambda(\mu_2). \end{cases}$$

Proof. The proposition can be proved by a standard transfinite induction on $\nu = \max\{\mu_1, \mu_2\}$.

Connect with Proposition 2.1, we ask the following question.

Question 2.1 Does there exsit a separable metrizable space X_α such that $\mathcal{A}(\alpha) \cap \mathcal{M}(\alpha)\text{-trind}\, X_\alpha > \max\{\mathcal{A}(\alpha)\text{-trind}\, X_\alpha, \mathcal{M}(\alpha)\text{-trind}\, X_\alpha\}$ for each ordinal number α? In particular, does there exist a separable metrizable space X such that $\mathcal{C}_0 \cap \mathcal{S}\text{-ind}\, X = 1$ and $\mathcal{C}_0\text{-ind}\, X = \mathcal{S}\text{-trind}\, X = 0$?

Recall from M.G. Charalambous ([2]) that we call a subset A of a space X *a Bernstein set* if $|A \cap B| = |(X \setminus A) \cap B| = c$ for every uncountable Borel set B of X, where c denotes the casrdinality of the continuum. It is known that every uncountable completely metrizable space X has countably many disjoint Berstein sets. We notice that $A \notin \mathcal{AB}$ for every Berstein set A of an uncountable completely metrizable space X.

Proof of Theorem 1.2. Let F be a Berstein set of $\mathbb{I}$. We put $X = (\{0\} \times \mathbb{I}^\infty) \cup (\bigcup_{n=1}^\infty \{\frac{1}{n}\} \times \pi_n^{-1}(F))$. Then, we can show that X is the desired space by an argument similar to Theorem 1.1.

Connect with Theorem 1.1, we may ask the following question.

Question 2.2 For each ordinal numbers α and β with $1 \leq \alpha < \omega_1$ and $0 \leq \beta < \omega_1$ do there exist separable metrizable spaces $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ which satisfy the folowing conditions?

1. $\mathcal{A}(\alpha)\text{-trind}\, X_{\alpha,\beta} = \beta$,
2. $\mathcal{M}(\alpha)\text{-trind}\, Y_{\alpha,\beta} = \beta$, and
3. $\mathcal{M}(\alpha)\text{-trind}\, X_{\alpha,\beta} = \mathcal{A}(\alpha)\text{-trind}\, Y_{\alpha,\beta} = -1$.

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